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BUCKLING AND STEADY FLUTTER OF HEAT-COMPRESSED PANELS IN A SUPERSONIC FLOW

V.V.Bolotin and Yu.N.Novichkov*

A systematic study of thermal buckling and steady flutter of panels, originally plane and rectangular in planform, based on a previous method for solving nonlinear stability problems of plane panels at high supersonic velocities is given. The form of elastic equilibrium of thermally buckled panels is determined and their stability is investigated. The amplitudes of the steady flutter are found in second approximation by the method of the small parameter and by the method of harmonic balance; the values found are mutually comparable. The behavior of panels in a flow under simultaneous variation of flow velocity and panel temperature is discussed.

The nonlinear problem of the steady flutter of a plane rectangular plate in a supersonic flow was first considered by one of us (Bibl.1). We subsequently presented an approximate method of investigating nonlinear problems of stability and flutter of plates and shells in a flow of gas, allowing for the stresses resulting from aerodynamic heating. The thermal buckling and steady flutter of a plate under cylindrical flexure have also been considered (Bibl.5). The present article is devoted to a systematic study of thermal buckling and

* Institute of Mechanics, USSR Academy of Sciences, Moscow.

** Numbers in the margin indicate pagination in the original foreign text.

flutter of panels originally plane and rectangular in planform, based on the method previously given (Bibl.2 - 3).

1. Let us consider an elastic plate resting on a rectangular contour with sides a and b, around which flows - in the direction of the axis Ox - a supersonic gas stream with the unperturbed velocity U (Fig.1). The equations of

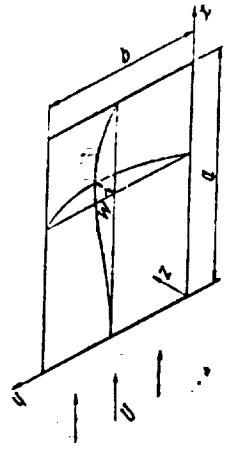

Fig.1

motion of this plate, allowing for the effect of thermal heating and aerodynamic forces, may be obtained as a special case of the equations (Bibl.2):

$$D\Delta\Delta w + \alpha(1+\nu)D\Delta\Theta + \rho_0 h\frac{\partial^2 w}{\partial t^2} + \rho_0 h\varepsilon\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 \Phi}{\partial x\,\partial y} + p_0 - p_\infty\left[1 + \frac{\varkappa-1}{2a_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right)\right]^{\frac{2\varkappa}{\varkappa-1}}; \quad (1.1)$$

$$\frac{1}{Eh}\Delta\Delta\Phi + \alpha\Delta T_0 = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \quad (1.2)$$

where w(x, y, t) is the deflection of the plate, Φ(x, y, t) a function of the stresses at the center surface, E the modulus of elasticity, ν Poisson's ratio, D the cylindrical stiffness, ρ_0 the density of the material of the plate, h the thickness of the plate, ε the structural damping factor, α the coefficient of thermal expansion, T_0(x, y, t) the temperature at the center surface, Θ(x, y, t)

the temperature gradient in the direction of the normal to the center surface. In this case, p_0 is the internal pressure, p_∞ the undisturbed external pressure, a_∞ the unperturbed speed of sound, and $\varkappa$ is the polytropic index of the gas. Equations (1.1) and (1.2) have been set up under the following assumptions: 1) Kirchhoff's hypothesis is valid; 2) the deflections of the plate are comparable with its thickness but small in comparison with its sides; 3) the tangential forces of inertia and damping are negligibly small; 4) the mechanical and thermophysical characteristics of the material are independent of temperature; 5) the aerodynamic forces may be found by the asymptotic piston theory. /83

The conditions along the contour for deflection w have the form

$$\left.\begin{aligned} w &= \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} + \alpha(1+\nu)\theta = 0 \quad \text{at} \quad x = 0 \text{ and } x = a \\ w &= \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} + \alpha(1+\nu)\theta = 0 \quad \text{at} \quad y = 0 \text{ and } y = b \end{aligned}\right\}. \tag{1.3}$$

Let us formulate the boundary conditions for the "in-the-mean" function (Bibl.2):

$$\overline{N}_x = c_x \overline{\Delta}_x, \quad \overline{N}_y = c_y \overline{\Delta}_y, \quad \overline{N}_{xy} = c_{xy} \overline{\Delta}_{xy} \tag{1.4}$$

where $\overline{N}_x$, $\overline{N}_y$, $\overline{N}_{xy}$ are the averaged stresses along the edges

$$\overline{N}_x = \frac{1}{b}\int_0^b \frac{\partial^2 \Phi(0, y)}{\partial y^2}\, dy, \quad \overline{N}_y = \frac{1}{a}\int_0^a \frac{\partial^2 \Phi(x, 0)}{\partial x^2} dx, \quad \overline{N}_{xy} = -\frac{1}{a}\int_0^a \frac{\partial^2 \Phi(x, 0)}{\partial x\, \partial y}\, dx, \tag{1.5}$$

while $\overline{\Delta}_x$, $\overline{\Delta}_y$, $\overline{\Delta}_{xy}$ are the average relative approaches of the edges and the relative shift

$$\begin{aligned} \overline{\Delta}_x &= -\frac{1}{ab}\int_0^b dy \int_0^a \frac{\partial u}{\partial x} dx, \quad \overline{\Delta}_y = -\frac{1}{ab}\int_0^a dx \int_0^b \frac{\partial v}{\partial y} dy, \\ \overline{\Delta}_{xy} &= -\frac{1}{2ab}\int_0^a dx \int_0^b \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dy, \end{aligned} \tag{1.6}$$

and c_x, c_y, and c_{xy} are certain coefficients characterizing the resistance of the reinforcing structure to tangential deformation of the plate; u and v are the tangential displacements of the points of the center surface, determined

from the relation

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}=\frac{1}{Eh}(N_x-\nu N_y)+\alpha T_0-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\
&\frac{\partial v}{\partial y}=\frac{1}{Eh}(N_y-\nu N_x)+\alpha T_0-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\
&\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\frac{1+\nu}{E}N_{xy}-\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\\
&N_x=\frac{\partial^2\Phi}{\partial y^2},\quad N_y=\frac{\partial^2\Phi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\Phi}{\partial x\,\partial y}
\end{aligned}\right\}. \tag{1.7}$$

The problem reduces to finding the solutions of eqs.(1.1) and (1.2) satisfying the boundary conditions (1.3) and (1.4) and also the assigned initial conditions. This problem is very complex and does not admit of an exact solution. An effective approximate equation can be constructed if the problem is reduced to a system of ordinary differential equations with respect to certain functions of time (Bibl.2).

We shall seek the approximate solution for w in the form of a series in the functions $\varphi_j(x, y)$, satisfying the conditions (1.3):

$$w(x, y, t)=\sum_{j=1}^{n} f_j(t)\,\varphi_j(x, y). \tag{1.8}$$

Substituting eq.(1.8) into the right-hand side of eq.(1.2), we obtain an equation linear in Φ. Let us assume that we were able to find a particular solution $\Phi_1(x, y, t)$ of this equation, having the property that the corresponding stresses (1.5) vanish. The expression /84

$$\Phi(x, y, t)=\Phi_1(x, y, t)+\frac{1}{2}(N_x^0y^2+N_y^0x^2-2N_{xy}^0xy). \tag{1.9}$$

where N_x^0, N_y^0, N_{xy}^0 are functions only of time, likewise satisfy the equation. Finding N_x^0, N_y^0, and N_{xy}^0 (the averaged stresses on the edges) from the conditions (1.4), let us substitute eqs.(1.8) and (1.9) into eq.(1.1), and require it to be approximately satisfied, in the sense of the Galerkin calculus of variations. The result is a system of n ordinary differential equations with respect to the functions $f_j(t)$.

2. Let us assume that $T_0 = T_0(t)$ is a slowly varying function of time and is independent of the coordinates x and y, and that $\varepsilon = 0$. Let us linearize the aerodynamic term, while the aerodynamic damping, since it does not depend on U, is included in ε. Let us also assume that $p_0 = p_\infty$.

The boundary conditions (1.3) will be satisfied in this case, if we assume that

$$\varphi_j = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (m, n = 1, 2, \ldots).$$

To obtain an approximate solution let us take only the first two terms:

$$w(x, y; t) = f_1(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + f_2(t)\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}. \tag{2.1}$$

Then, the particular solution of eq.(1.2) may be represented in the form

$$\begin{aligned}\Phi_1 = \frac{Ehm^2}{4}\Big[&- f_1 f_2\cos\frac{\pi x}{a} + \frac{1}{8}f_1^2\cos\frac{2\pi x}{a} + \frac{1}{9}f_1 f_2\cos\frac{3\pi x}{a} + \\ &+ \frac{1}{32}f_2^2\cos\frac{4\pi x}{a} + \frac{1}{8m^4}(f_1^2 + f_2^2)\cos\frac{2\pi y}{b} + \frac{9f_1 f_2}{(1+4m^2)^2}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b} - \\ &- \frac{f_1 f_2}{(9+4m^2)^2}\cos\frac{3\pi x}{a}\cos\frac{2\pi y}{b}\Big],\end{aligned} \tag{2.2}$$

where m = a/b. Substituting eqs.(2.1) and (1.9) into eqs.(1.4) - (1.7), we obtain

$$\left.\begin{aligned}N_x^0 &= \frac{\pi^2 Eh\beta_x}{8a^2(1-\nu^2\beta_x\beta_y)}\Big[-\frac{8a^2(1+\nu\beta_y)}{\pi^2}\alpha T + (f_1^2 + 4f_2^2) + m^2\beta_y\nu(f_1^2 + f_2^2)\Big] \\ N_y^0 &= \frac{\pi^2 Eh\beta_y}{8a^2(1-\nu^2\beta_x\beta_y)}\Big[-\frac{8a^2(1+\nu\beta_x)}{\pi^2}\alpha T + \nu\beta_x(f_1^2 + 4f_2^2) + m^2(f_1^2 + f_2^2)\Big]\end{aligned}\right\}, \tag{2.3}$$

where

$$\frac{1}{\beta_x} = 1 + \frac{Eh}{c_x}, \quad \frac{1}{\beta_y} = 1 + \frac{Eh}{c_y}. \tag{2.4}$$

Let us equate the stress N_{xy}^0 to zero, which is equivalent to the assumption that the reinforcing structure does not offer resistance to the skewing of the contour in the plane ($c_{xy} = 0$).

The Galerkin method, recalling eqs.(1.9), and (2.1) - (2.4), leads to the following system of equations:

$$\left.\begin{aligned}\varphi_1'' + g\varphi_1' + \Omega_1^2\varphi_1 - \mu\varphi_2 + c_{11}\varphi_1^3 + c_{12}\varphi_1\varphi_2^2 = 0 \\ \varphi_2'' + g\varphi' + \Omega_2^2\varphi_2 + \mu\varphi_1 + c_{21}\varphi_1^2\varphi_2 + c_{22}\varphi_2^3 = 0\end{aligned}\right\}. \tag{2.5}$$

Here the primes denote differentiation with respect to dimensionless time $\tau = \Omega t$, and the following notation has been introduced: /85

$$\left.\begin{aligned}
&\varphi_1 = \frac{J_1}{h}, \quad \varphi_2 = \frac{J_2}{h}, \quad g = \frac{\varepsilon}{\Omega}, \quad \Omega^2 = \frac{D\pi^4}{\rho_0 h a^4} \\
&\Omega_1^2 = (1+m^2)^2 - \frac{12(1-\nu^2)a^2\alpha T}{\pi^2 h^2(1-\nu^2\beta_x\beta_y)}[\beta_x(1+\nu\beta_y) + m^2\beta_y(1+\nu\beta_x)] \\
&\Omega_2^2 = (4+m^2)^2 - \frac{12(1-\nu^2)a^2\alpha T}{\pi^2 h^2(1-\nu^2\beta_x\beta_y)}[4\beta_x(1+\nu\beta_y) + m^2\beta_y(1+\nu\beta_x)] \\
&\mu = \frac{8}{3}\frac{\varkappa p_\infty a^3 U}{D\pi^4 a_\infty} \\
&c_{11} = \frac{3}{4}(1-\nu^2)\left[1 + m^4 + \frac{2(\beta_x + 2m^2\nu\beta_x\beta_y + m^4\beta_y)}{1-\nu^2\beta_x\beta_y}\right] \\
&c_{12} = c_{21} = \frac{3}{4}(1-\nu^2)\left[4(1+m^4) + \frac{81m^4}{(1+4m^2)^2} + \frac{m^4}{(9+4m^2)^2} + \right. \\
&\qquad\qquad \left. + \frac{2(4\beta_x + 5m^2\nu\beta_x\beta_y + m^4\beta_y)}{1-\nu^2\beta_x\beta_y}\right] \\
&c_{22} = \frac{3}{4}(1-\nu^2)\left[16 + m^4 + \frac{2(16\beta_x + 8m^2\nu\beta_x\beta_y + m^4\beta_y)}{1-\nu^2\beta_x\beta_y}\right]
\end{aligned}\right\}. \qquad (2.6)$$

3. The system (2.5) admits of the trivial solution $\omega_1 = \omega_2 = 0$, corresponding to the plane form of equilibrium of the plate. Let us investigate the stability of this solution. Substituting ω_1 and ω_2 in eqs.(2.5) by $\delta\omega_1$ and $\delta\omega_2$, and linearizing in $\delta\omega_1$ and $\delta\omega_2$, we obtain

$$\left.\begin{aligned}
&\delta\varphi_1'' + g\delta\varphi_1' + \Omega_1^2\delta\varphi_1 - \mu\delta\varphi_2 = 0 \\
&\delta\varphi_2'' + g\delta\varphi_2' + \Omega_2^2\delta\varphi_2 + \mu\delta\varphi_1 = 0
\end{aligned}\right\}. \qquad (3.1)$$

Finding the solution of these equations in the form

$$\delta\varphi_1 = C_1 e^{\lambda\tau}, \quad \delta\varphi_2 = C_2 e^{\lambda\tau} \qquad (3.2)$$

and writing the existence condition of a nontrivial solution for C_1 and C_2, we arrive at the characteristic equation

$$\begin{vmatrix} \lambda^2 + g\lambda + \Omega_1^2 & -\mu \\ \mu & \lambda^2 + g\lambda + \Omega_2^2 \end{vmatrix} = 0. \qquad (3.3)$$

Now let

$$\left.\begin{aligned} \theta &= \frac{12\alpha T(1+\nu)a^3}{\pi^3 h^3}, \quad \omega_{jk}^2 = j^2 + m^2k^2 \\ s_{jk} &= \frac{j^2\beta_x(1+\nu\beta_y) + k^2m^2(1+\nu\beta_x\beta_y)}{1+\nu} \cdot \frac{1-\nu^2}{1-\nu^2\beta_x\beta_y} \end{aligned}\right\}, \tag{3.4}$$

$$\Omega_j^2 = \omega_{j1}^2 - \theta s_{j1}.$$

Then, the characteristic equation takes the form

$$\begin{aligned} &\lambda^4 + 2g\lambda^3 + [\omega_{11}^2 + \omega_{21}^2 + g^2 - \theta(s_{11}+s_{21})]\lambda^2 + \\ &+ g[\omega_{11}^2 + \omega_{21}^2 - \theta(s_{11}+s_{21})]\lambda + (\omega_{11}^2 - \theta s_{11})(\omega_{21}^2 - \theta s_{21}) + \mu^2 = 0. \end{aligned} \tag{3.5}$$

The trivial solution will be stable if all the roots of the characteristic equation (3.5) lie in the left half-plane of the complex variable, i.e., if their real parts are negative. For this, all the coefficients and the Routh determinant must be positive. For the equation /86

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

there must be

$$\left.\begin{aligned} &a_3 > 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 > 0 \\ &R = a_1a_2a_3 - a_0a_3^2 - a_1^2 > 0 \end{aligned}\right\}. \tag{3.6}$$

The requirements that the free term and the Routh determinant R must be positive are essential in this case. The boundaries of stability are obtained by equating a_0 and R to zero. The first condition

$$(\omega_{11}^2 - \theta s_{11})(\omega_{21}^2 - \theta s_{21}) + \mu^2 = 0 \tag{3.7}$$

in the plane (ω, θ) defines an ellipse. The second condition

$$2g^2[\omega_{11}^2 + \omega_{21}^2 - \theta(s_{11}+s_{21})] + [\omega_{21}^2 - \omega_{11}^2 - \theta(s_{21}-s_{11})]^2 - 4\mu^2 = 0 \tag{3.8}$$

defines two intersecting straight lines in the same plane [if we pose $g^2 \ll$ $\ll |\Omega_2^2 - \Omega_1^2|$, considering the damping to be sufficiently small, and the partial frequencies not too close together], but interest attaches only to the line with a negative angular coefficient. It is easy to show that this line and the ellipse osculate at the point with the coefficients

$$\mu = \frac{\omega_{21}^2 s_{11} - \omega_{11}^2 s_{21}}{s_{11} + s_{21}}, \quad \theta = \frac{\omega_{11}^2 + \omega_{21}^2}{s_{11} + s_{21}}. \tag{3.9}$$

It is not difficult to obtain, for the roots of eq.(3.5), the explicit expression:

$$\lambda = -\frac{g}{2} \pm \left\{ \frac{g^2}{4} - \frac{1}{2}\left[\omega_{11}^2 + \omega_{21}^2 - \theta (s_{11} + s_{21}) \pm \right.\right.$$
$$\left.\left. \pm \left([\omega_{21}^2 - \omega_{11}^2 - \theta (s_{21} - s_{11})]^2 - 4\mu^2 \right)^{1/2} \right] \right\}^{1/2}. \tag{3.10}$$

This expression gives a complete picture of the behavior of the roots of the characteristic equation. In the region bounded by the axes of coordinates,

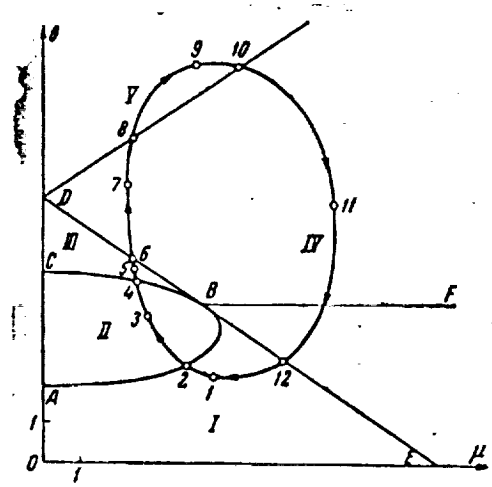

Fig.2

by the part of the ellipse AB, and by the straight line BE (region I, Fig.2), all the roots have negative real parts. Consequently, this region is the stability region of the trivial solution of the system (2.5), i.e., the stability region of the planform of the plate. Departure from this region through the ellipse corresponds to a nonoscillatory stability loss (one of the roots passes into the right half-plane through the origin of coordinates). On the other hand, its deviation across the straight line corresponds to an oscillatory stability loss (the two roots pass into the right-hand half-plane, intersecting the axis of imaginaries). In the other regions the trivial solution is unstable,

with varying degree of instability; in the region II the degree of instability is 1, and in the others it is 2 (in the regions III and V the character of the roots is the same, but differs from the character of the roots in the region IV). For clarity, Fig.3 shows the pattern of variation in the character of the /87 roots in the plane of the complex variable in the circulation about a certain closed contour in the plane (u, θ); see Fig.2.

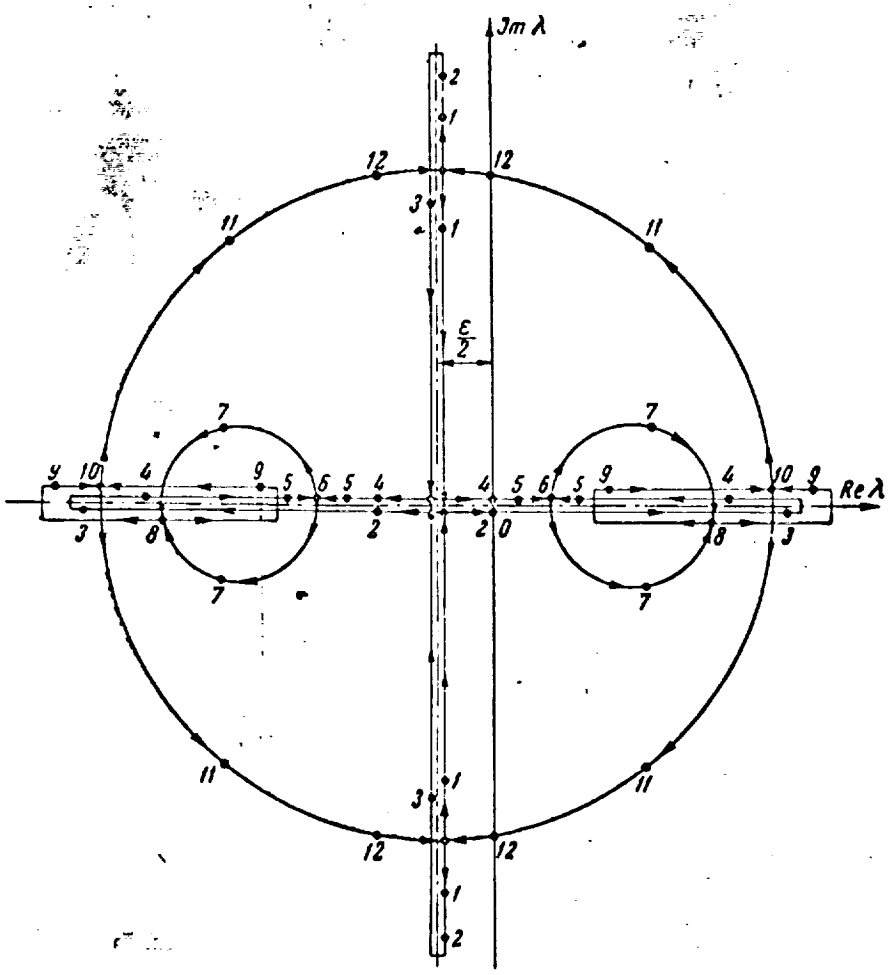

Fig.3

4. The time-invariant nontrivial solutions of the system (2.5) correspond to the static forms of thermal buckling. Let us investigate these forms. We will start from the equations

$$\left.\begin{aligned}(\omega_{11}^2 - 0s_{11})\,\varphi_1 - \mu\varphi_2 + c_{11}\varphi_1^3 + c_{12}\varphi_1\varphi_2^2 = 0\\ (\omega_{21}^2 - 0s_{21})\,\varphi_2 + \mu\varphi_1 + c_{21}\varphi_1^2\varphi_2 + c_{22}\varphi_2^3 = 0\end{aligned}\right\}. \qquad (4.1)$$

Equations (4.1) are linear in the parameters μ and θ, so that the inverse method of solution becomes convenient:

$$\left.\begin{aligned}0 &= \frac{c_{11}\varphi_1^4 + 2c_{12}\varphi_1^2\varphi_2^2 + c_{22}\varphi_2^4 + \omega_{11}^2\varphi_1^2 + \omega_{21}^2\varphi_2^2}{s_{11}\varphi_1^2 + s_{21}\varphi_2^2}\\ \mu &= \frac{(s_{21}c_{11} - s_{11}c_{21})\,\varphi_1^3\varphi_2 + (s_{21}c_{12} - s_{11}c_{22})\,\varphi_1\varphi_2^3 + (s_{21}\omega_{11}^2 - s_{11}\omega_{21}^2)\,\varphi_1\varphi_2}{s_{11}\varphi_1^2 + s_{21}\varphi_2^2}\end{aligned}\right\}. \qquad (4.2)$$

In the plane (μ, θ), eqs.(4.2) represent two families of curves φ_1 = const and φ_2 = const (Fig.4). These curves are horizontals of certain surfaces, each of which is the projection of a certain four-dimensional surface onto three- /88
dimensional spaces: the first family, onto the space (μ, θ, φ_1) and the second family, onto the space (μ, θ, φ_2).

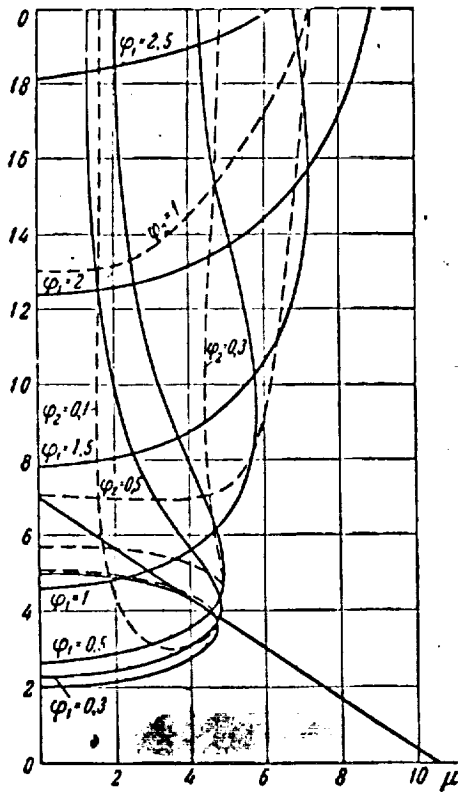

Fig.4

5. To investigate the stability of the solutions obtained for the system of equations (2.5), let us substitute ω_1 and ω_2 by $\omega_1 + \delta\omega_1$ and $\omega_2 + \delta\omega_2$ respectively, and let us linearize the equations with respect to the variations of $\delta\omega_1$ and $\delta\omega_2$. Bearing in mind that ω_1 and ω_2 are stationary in this case and satisfy eqs.(4.1), we obtain

$$\left.\begin{aligned}\delta\varphi_1'' + g\delta\varphi_1' + (\omega_{11}^2 - 0s_{11} + 3c_{11}\varphi_1^2 + c_{12}\varphi_2^2)\,\delta\varphi_1 + (2c_{12}\varphi_1\varphi_2 - \mu)\,\delta\varphi_2 = 0\\ \delta\varphi_2'' + g\delta\varphi_2' + (\omega_{21}^2 - 0s_{21} + 3c_{22}\varphi_2^2 + c_{21}\varphi_1^2)\,\delta\varphi_2 + (2c_{21}\varphi_1\varphi_2 + \mu)\,\delta\varphi_1 = 0\end{aligned}\right\}. \qquad (5.1)$$

Let us seek the solution as before in the form of eqs.(3.2). We arrive at the characteristic equation

$$\begin{vmatrix}\lambda^2 + g\lambda + \omega_{11}^2 - 0s_{11} + 3c_{11}\varphi_1^2 + c_{12}\varphi_2^2 & 2c_{12}\varphi_1\varphi_2 - \mu\\ 2c_{21}\varphi_1\varphi_2 + \mu & \lambda^2 + g\lambda + \omega_{21}^2 - 0s_{21} + 3c_{22}\varphi_2^2 + c_{21}\varphi_1^2\end{vmatrix} = 0. \qquad (5.2)$$

Applying the Routh-Hurwitz criterion (3.6), we arrive in this case at

$$\left.\begin{aligned}a_0 &= (\omega_{11}^2 - 0s_{11} + 3c_{11}\varphi_1^2 + c_{12}\varphi_2^2)(\omega_{21}^2 - 0s_{21} + 3c_{22}\varphi_2^2 + c_{21}\varphi_1^2) -\\ &\quad - (2c_{12}\varphi_1\varphi_2 - \mu)(2c_{21}\varphi_1\varphi_2 + \mu)\\ a_1 &= g\left[\omega_{11}^2 + \omega_{21}^2 - 0(s_{11} + s_{21}) + (3c_{11} + c_{21})\varphi_1^2 + (c_{12} + 3c_{22})\varphi_2^2\right]\\ a_2 &= g^2 + \omega_{11}^2 + \omega_{21}^2 - 0(s_{11} + s_{21}) + (3c_{11} + c_{21})\varphi_1^2 + (c_{12} + 3c_{22})\varphi_2^2\\ a_3 &= 2g\end{aligned}\right\}. \qquad (5.3)$$

The resolvent, from the conditions (3.6), is $a_0 = 0$. We will demonstrate that the line, along which $a_0 = 0$, coincides with the common envelope of the /89 families of curves $\omega_1 = \text{const}$ and $\omega_2 = \text{const}$ (Fig.4). Let us write eq.(4.1) in the form of

$$F_1(\varphi_1, \varphi_2, 0, \mu) = 0, \quad F_2(\varphi_1, \varphi_2, 0, \mu) = 0. \qquad (5.4)$$

To obtain the equation of the envelope of these families, we must eliminate the parameters ω_1 and ω_2 from eqs.(5.4) and make use of the condition

$$\frac{\partial F_1}{\partial\varphi_1}\frac{\partial F_2}{\partial\varphi_2} - \frac{\partial F_1}{\partial\varphi_2}\frac{\partial F_2}{\partial\varphi_1} = 0. \qquad (5.5)$$

It is not difficult to see that the condition (5.5), if we consider that ω_1 and ω_2 have already been eliminated by the aid of eqs.(5.4), coincides with

the condition $a_0 = 0$.

6. We will seek the periodic solutions of the system (2.5) corresponding to steady flutter of the panel. In the neighborhood of the boundary of the region of flutter BE we shall seek the solution of the system (2.5) by the method of the small parameter. A first approximation has already been constructed by this method (Bibl.2). We shall now obtain a second approximation by the method of the small parameter and shall thus give an estimate of the error of the formulas derived earlier by us in (Bibl.2).

Considering the neighborhood of the critical value, let us put

$$\mu = \mu_* + \eta\mu_1, \tag{6.1}$$

where η is a small parameter and μ_* is defined by the relation

$$\mu_* = \frac{1}{2}\{[\omega_{11}^2 - \omega_{11}^2 - \theta(s_{21} - s_{11})]^2 + 2g^2[\omega_{11}^2 + \omega_{11}^2 - \theta(s_{21} + s_{11})]\}^{1/2}. \tag{6.2}$$

As the generating system let us take

$$\left.\begin{aligned} \varphi_1'' + g\varphi_1' + \Omega_1^2\varphi_1 - \mu_*\varphi_2 = 0 \\ \varphi_2'' + g\varphi_2' + \Omega_2^2\varphi_2 + \mu_*\varphi_1 = 0 \end{aligned}\right\}, \tag{6.3}$$

which permits of two linearly independent solutions $\varphi_{j1}(\tau)$ and $\varphi_{j2}(\tau)$, corresponding to the values $\lambda = \pm i\omega_*$, where

$$\omega_* = \left(\frac{\Omega_1^2 + \Omega_2^2}{2}\right)^{1/2}. \tag{6.4}$$

In what follows, let us take $g^2 < |\Omega_2^2 - \Omega_1^2|$. Then, we have approximately

$$\mu_* = \frac{\Omega_2^2 - \Omega_1^2}{2} \cong \omega_*^2 - \Omega_1^2 = \Omega_2^2 - \omega_*^2. \tag{6.5}$$

Let us introduce the small parameter into eqs.(2.5) in the following manner:

$$\left.\begin{aligned} \varphi_1'' + g\varphi_1' + \Omega_1^2\varphi_1 - \mu_*\varphi_2 = \eta\mu_1\varphi_2 - \eta(c_{11}\varphi_1^2 + c_{12}\varphi_1\varphi_2^2) \\ \varphi_2'' + g\varphi_2' + \Omega_2\varphi_2 + \mu_*\varphi_1 = -\eta\mu_1\varphi_1 - \eta(c_{21}\varphi_1^2\varphi_2 + c_{22}\varphi_2^3) \end{aligned}\right\}. \tag{6.6}$$

Let us seek the solution of this system in the form of

$$\left.\begin{aligned}\varphi_1(\tau_1) &= A\varphi_{11}(\tau_1) + \sum_{k=1}^{\infty}\eta^k\varphi_1^{(k)}(\tau_1)\\ \varphi_2(\tau_1) &= A\varphi_{21}(\tau_1) + \sum_{k=1}^{\infty}\eta^k\varphi_2^{(k)}(\tau_1)\end{aligned}\right\}, \tag{6.7}$$

where A is the amplitude of the first approximation, and

$$\tau_1 = \omega_*\tau(1 + \eta p_1 + \eta^2 p_2 + \ldots). \tag{6.8}$$

Here p_i are the corrections to the frequency of the oscillations.

Substituting eqs.(6.7) and (6.8) into eqs.(6.6), and equating the coef- /90
ficients of the same powers of η, we arrive at the following equations:

$$\left.\begin{aligned}\omega_*^2\ddot{\varphi}_{11} + g\omega_*\dot{\varphi}_{11} + \Omega_1^2\varphi_{11} - \mu_*\varphi_{21} &= 0\\ \omega_*^2\ddot{\varphi}_{21} + g\omega_*\dot{\varphi}_{21} + \Omega_2^2\varphi_{21} + \mu_*\varphi_{11} &= 0\end{aligned}\right\} \tag{6.9}$$

$$\left.\begin{aligned}\omega_*^2\ddot{\varphi}_1^{(k)} + g\omega_*\dot{\varphi}_1^{(k)} + \Omega_1^2\varphi_1^{(k)} - \mu_*\varphi_2^{(k)} &= \Psi_1^{(k)}\\ \omega_*^2\ddot{\varphi}_2^{(k)} + g\omega_*\dot{\varphi}_2^{(k)} + \Omega_2^2\varphi_2^{(k)} + \mu_*\varphi_1^{(k)} &= \Psi_2^{(k)}\end{aligned}\right\} \quad (k = 1, 2, \ldots) \tag{6.10}$$

In all these equations, the dot over the φ indicates differentiation with respect to τ_1; let us also write out the formulas for $\Psi_1^{(1)}$, $\Psi_2^{(1)}$, $\Psi_1^{(2)}$, and $\Psi_2^{(2)}$:

$$\left.\begin{aligned}\Psi_1^{(1)} &= -2\omega_*^2 p_1 A\ddot{\varphi}_{11} - g\omega_* p_1 A\dot{\varphi}_{11} + \mu_1 A\varphi_{21} - c_{11}A^3\varphi_{11}^3 - c_{12}A^3\varphi_{11}\varphi_{21}^2\\ \Psi_2^{(1)} &= -2\omega_*^2 p_1 A\ddot{\varphi}_{21} - g\omega_* p_1 A\dot{\varphi}_{21} - \mu_1 A\varphi_{11} - c_{21}A^3\varphi_{11}^2\varphi_{21} - c_{22}A^3\varphi_{21}^3\end{aligned}\right\} \tag{6.11}$$

$$\left.\begin{aligned}\Psi_1^{(2)} &= -\omega_*^2 p_1^2 A\ddot{\varphi}_{11} - 2\omega_*^2 p_2 A\ddot{\varphi}_{11} - 2\omega_*^2 p_1\ddot{\varphi}_1^{(1)} - g\omega_* p_2 A\dot{\varphi}_{11} -\\ &\quad - g\omega_* p_1\dot{\varphi}_1^{(1)} + \mu_1\varphi_2^{(1)} - 3c_{11}A^2\varphi_{11}^2\varphi_1^{(1)} -\\ &\quad - c_{12}A^2[\varphi_{21}^2\varphi_1^{(1)} + 2\varphi_{11}\varphi_{21}\varphi_2^{(1)}]\\ \Psi_2^{(2)} &= -\omega_*^2 p_1^2 A\ddot{\varphi}_{21} - 2\omega_*^2 p_2 A\ddot{\varphi}_{21} - 2\omega_*^2 p_1\ddot{\varphi}_2^{(1)} - g\omega_* p_2 A\dot{\varphi}_{21} -\\ &\quad - g\omega_* p_1\dot{\varphi}_2^{(1)} - \mu_1\varphi_1^{(1)} - c_{21}A^2[\varphi_{11}^2\varphi_2^{(1)} +\\ &\quad + 2\varphi_{11}\varphi_{21}\varphi_1^{(1)}] - 3c_{22}A^2\varphi_{21}^2\varphi_2^{(1)}.\end{aligned}\right\} \tag{6.12}$$

7. Equations (6.9) are satisfied by the choice of the generating solutions, which can be easily found; in the real form, for $g^2 \ll |\Omega_2^2 - \Omega_1^2|$, they will be

$$\left.\begin{aligned}\varphi_{11} &= \cos\tau_1 + \ldots\\ \varphi_{21} &= -\cos\tau_1 - g\chi\sin\tau_1 + \ldots\\ \varphi_{12} &= \sin\tau_1 + \ldots\\ \varphi_{22} &= -\sin\tau_1 + g\chi\cos\tau_1 + \ldots\end{aligned}\right\} \tag{7.1}$$

where

$$\chi = \frac{\omega_*}{\mu_*} = \frac{\sqrt{2}\sqrt{\Omega_1^2 + \Omega_2^2}}{\Omega_2^2 - \Omega_1^2}. \tag{7.2}$$

Let us find the amplitude of the first approximation A and the correction to the frequency p_1 from the stipulation that the system (6.10), for k = 1, have periodic solutions. As previously shown (Bibl.2), for the system (6.10) the condition of periodicity has the form

$$\left.\begin{aligned} P_1^{(s)} + P_2^{(s)} - g\chi Q_2^{(s)} = 0 \\ Q_1^{(s)} + Q_2^{(s)} + g\chi P_2^{(s)} = 0 \end{aligned}\right\} \tag{7.3}$$

in which $P_i^{(s)}$, $Q_i^{(s)}$ are determined from the expansions in Fourier series of the functions

$$\begin{aligned} \Psi_i^{(s)} &= P_i^{(s)} \cos\tau_1 + Q_i^{(s)} \sin\tau_1 + \dots \\ i &= 1, 2; \quad s = 1, 2, 3 \dots \end{aligned} \tag{7.4}$$

Substituting eqs.(7.1) into eqs.(6.11), we obtain /01

$$\left.\begin{aligned} P_1^{(1)} &= 2\omega_*^2 p_1 A - \mu_1 A - \frac{3}{4} c_{11} A^3 - \frac{3}{4} c_{12} A^3 \\ Q_1^{(1)} &= g\omega_* p_1 A - g\chi\mu_1 A - \frac{1}{2} g\chi c_{12} A^3 \\ P_2^{(1)} &= -2\omega_*^2 p_1 A - \mu_1 A + \frac{3}{4} c_{12} A^3 + \frac{3}{4} c_{22} A^3 \\ Q_2^{(1)} &= -2g\chi\omega_*^2 p_1 A - g\omega_* p_1 A + \frac{1}{4} g\chi c_{21} A^3 + \frac{3}{4} g\chi c_{22} A^3 \end{aligned}\right\} \tag{7.5}$$

At small damping, the conditions (7.3) yield

$$A = \left(\frac{8}{3}\frac{\mu - \mu_*}{c_0}\right)^{1/2}; \quad p_1 = \frac{1}{16}\frac{A^2}{\omega_*^2} c_1, \tag{7.6}$$

where

$$c_0 = c_{22} - c_{11}, \quad c_1 = 3c_{11} + 2c_{12} + 3c_{22}. \tag{7.7}$$

We note that $c_0 > 0$ and $c_1 > 0$. The first of eqs.(7.6) coincides with the corresponding formula in our earlier paper (Bibl.2) where, in contrast to the present work, the aerodynamic nonlinearity is taken into account.

Let us then pass to the determination of the second approximation. We will seek the functions $v_1^{(1)}$ and $v_2^{(1)}$ in the form of

$$\left.\begin{aligned}\varphi_1^{(1)} &= A_1\varphi_{11}(\tau_1) + \xi_1(\tau_1)\\ \varphi_2^{(1)} &= A_1\varphi_{21}(\tau_1) + \xi_2(\tau_1)\end{aligned}\right\}. \tag{7.8}$$

Here, A_1 is the correction to the amplitude of the generating functions, while ξ_1 and ξ_2 is any particular solution of the system (6.10) at k = 1, when the values of A and p_1 from eq.(7.6) have been substituted into the right-hand sides of this system:

$$\left.\begin{aligned}&\omega_*^2\ddot{\xi}_1 + g\omega_*\dot{\xi}_1 + \Omega_1^2\xi_1 - \mu_*\xi_2 = -\frac{A^3c_{13}}{2}\cos\tau_1 -\\ &-g\chi A^3\left(d_1 + \frac{1}{2}c_{12}\right)\sin\tau_1 - \frac{A^3}{4}(c_{11}+c_{12})\cos 3\tau_1 -\\ &-\frac{A^3c_{13}}{2}g\chi\sin 3\tau_1\\ &\omega_*^2\ddot{\xi}_2 + g\omega_*\dot{\xi}_2 + \Omega_1^2\xi_2 + \mu_*\xi_1 = \frac{A^3c_{13}}{2}\cos\tau_1 +\\ &+g\chi A^3 d_1\sin\tau_1 + \frac{A^3}{4}(c_{21}+c_{22})\cos 3\tau_1 +\\ &+\frac{A^3}{4}g\chi(c_{21}+3c_{22})\sin 3\tau_1,\end{aligned}\right\} \tag{7.9}$$

where

$$d_1 = \frac{1}{16}\left(6c_0 - \frac{\mu_*}{\omega_*^2}c_1\right). \tag{7.10}$$

Let us find the solution of this system by the method of indeterminate coefficients, taking for ξ_1 and ξ_2 the expressions

$$\left.\begin{aligned}\xi_1 &= A^3[M_{11}\cos\tau_1 + g\chi M_{12}\sin\tau_1 + M_{13}\cos 3\tau_1 + g\chi M_{14}\sin 3\tau_1]\\ \xi_2 &= A^3[M_{21}\cos\tau_1 + g\chi M_{22}\sin\tau_1 + M_{23}\cos 3\tau_1 + g\chi M_{24}\sin 3\tau_1]\end{aligned}\right\} \tag{7.11}$$

with the indeterminate coefficients M_{ij} (i = 1, 2; j = 1, 2, 3, 4).

Substituting eqs.(7.11) into eqs.(7.9), equating the coefficients of cos τ_1, sin τ_1, cos $3\tau_1$, and sin $3\tau_1$, and neglecting terms with g in the second or higher powers, we arrive at the system of linear equations: /92

$$\left.\begin{aligned}&-\omega_*^2M_{11} + \Omega_1^2M_{11} - \mu_*M_{21} = -\frac{c_{13}}{2}\\ &-\omega_*^2M_{12} + \Omega_1^2M_{12} - \mu_*M_{11} - \mu_*M_{22} = -\left(d_1 + \frac{1}{2}c_{12}\right)\\ &-\omega_*^2M_{21} + \Omega_1^2M_{21} + \mu_*M_{11} = \frac{c_{13}}{2}\\ &-\omega_*^2M_{22} + \Omega_1^2M_{22} - \mu_*M_{21} + \mu_*M_{12} = d_1\end{aligned}\right| \tag{7.12}$$

$$
\left.
\begin{aligned}
&-9\omega_*^2 M_{13} + \Omega_1^2 M_{13} - \mu_* M_{23} = -\frac{1}{4}(c_{11} + c_{12}) \\
&-9\omega_*^2 M_{14} + \Omega_1^2 M_{14} - 3\mu_* M_{13} - \mu_* M_{24} = -\frac{c_{12}}{2} \\
&-9\omega_*^2 M_{23} + \Omega_1^2 M_{23} + \mu_* M_{13} = \frac{1}{4}(c_{21} + c_{22}) \\
&-9\omega_*^2 M_{24} + \Omega_1^2 M_{24} - 3\mu_* M_{23} + \mu_* M_{14} = \frac{1}{4}(c_{21} + 3c_{22})
\end{aligned}
\right|
$$

A particular solution of this system will be

$$
\left.
\begin{aligned}
&M_{12} = M_{22} = 0; \quad M_{11} = \frac{d_1}{\mu_*} + \frac{c_{12}}{2\mu_*}; \quad M_{21} = -\frac{d_1}{\mu_*} \\
&M_{13} = \frac{1}{32\omega_*^2}(c_{11} + c_{12}) + \frac{\mu_*}{256\omega_*^4} c_0; \\
&M_{23} = -\frac{1}{32\omega_*^2}(c_{22} + c_{21}) - \frac{\mu_*}{256\omega_*^4} c_0 \\
&M_{14} = \frac{c_{12}}{16\omega_*^2} - \frac{\mu_*}{256\omega_*^4}(3c_{11} + 4c_{12} - 3c_{22}) - \frac{3\mu_*^2}{1024\omega_*^6} c_0 \\
&M_{24} = -\frac{1}{32\omega_*^2}(c_{12} + 3c_{22}) + \frac{\mu_* c_{12}}{64\omega_*^4} + \frac{3\mu_*^2}{1024\omega_*^6} c_0
\end{aligned}
\right| \qquad (7.13)
$$

Making use of eqs.(7.1), (7.4), (7.6), and (7.8), into the last of which eqs.(7.11) have been introduced, and taking account of eqs.(7.13), we obtain

$$
\left.
\begin{aligned}
P_1^{(2)} &= \frac{A^6}{256\omega_*^2} c_1^2 + 2\omega_*^2 p_2 A - A^2 A_1 \left(\frac{3}{2} c_{11} + 2c_{12}\right) - \\
&- \frac{1}{8} A^6 M_{11}(15c_{11} + 4c_{12} - 3c_{22}) - \frac{1}{4} A^6 M_{13}(3c_{11} + c_{12}) + \\
&+ \frac{3}{8} A^6 M_{21}(c_0 + 4c_{12}) + \frac{1}{2} c_{12} A^6 M_{23} \\
Q_1^{(2)} &= g\omega_* p_2 A - g\chi A^2 A_1 \left(d_1 + \frac{3}{2} e_{12}\right) + g\chi A^6 M_{11} \left(\frac{\mu_*}{16\omega_*^2} c_1 - \right. \\
&\left. - \frac{1}{2} c_{12}\right) - \frac{3}{8} g\chi A^6 c_0 M_{13} + \frac{1}{2} c_{12} A^6 g\chi M_{13} - \\
&- \frac{1}{4} A^6 g\chi M_{14}(3c_{11} + c_{12}) + \frac{1}{2} c_{12} A^6 g\chi M_{21} + \\
&+ \frac{1}{8} A^6 g\chi M_{23}(3c_0 + 4c_{12}) - \frac{1}{2} c_{12} A^6 g\chi M_{23} + \frac{1}{2} c_{12} A^6 g\chi M_{24}
\end{aligned}
\right|
$$

$$
\left.
\begin{aligned}
P_2^{(2)} &= -\frac{A^6}{256\omega_*^2} c_1^2 - 2\omega_*^2 p_2 A + A^2 A_1 \left(2c_{12} + \frac{3}{2} c_{22}\right) - \\
&- \frac{3}{8} A^6 M_{11}(c_0 - 4c_{12}) + \frac{1}{2} c_{21} A^6 M_{13} - \\
&- \frac{1}{8} A^6 M_{21}(15c_{22} + 4c_{12} - 3c_{11}) - \frac{1}{4} A^6 M_{23}(3c_{22} + c_{12}) \\
Q_2^{(2)} &= -g\chi \frac{A^6}{256\omega_*^2} c_1^2 - g\omega_* p_2 A(1 + 2\chi\omega_*) +
\end{aligned}
\right| \qquad (7.14)
$$

/93

$$+ g\chi A^3 A_1\left[d_1 + \frac{1}{2}(c_{12} + 3c_{22})\right] + \frac{1}{2} c_{21} A^5 M_{11} g\chi -$$
$$- \frac{1}{8} g\chi A^5 M_{12}(3c_0 - 4c_{12}) - \frac{1}{2} c_{12} A^5 M_{12} g\chi +$$
$$+ \frac{1}{2} g\chi A^5 M_{14} c_{12} + g\chi A^5 M_{21}\left[\frac{\mu_*}{16\omega_*^2} c_1 - \frac{3}{2} c_{22}\right] -$$
$$- \frac{3}{8} c_0 g\chi A^5 M_{22} + \frac{3}{2} g\chi c_{22} A^5 M_{23} - \frac{1}{4} g\chi A^5 M_{24}(3c_{22} + c_{12}) \Bigg|$$

Applying the conditions of periodicity (7.3), we obtain for the correction to the amplitude of the generating solutions the expression:

$$A_1 = \left(\frac{8}{3}\frac{\mu - \mu_*}{c_0}\right)^{1/2}\left(\frac{\mu}{\mu_*} - 1\right)\left(-1 + \frac{4}{9}\frac{c_{12}(3c_{11} - 2c_{12})}{c_0^2} + \right.$$
$$\left. + \frac{11}{72}\frac{\mu_*}{\omega_*^2 c_0} c_1 - \frac{\mu_*^2}{192\omega_*^4}\right). \qquad (7.15)$$

To return to the initial problem, we must set $\eta = 1$. Then, we obtain, for the second approximation,

$$\varphi_1 = (A + A_1 + A^3 M_{11})\cos\tau_1 + g\chi A^3 M_{12}\sin\tau_1 +$$
$$+ A^3 M_{13}\cos 3\tau_1 + g\chi A^3 M_{14}\sin 3\tau_1;$$
$$\varphi_2 = (-A - A_1 + A^3 M_{21})\cos\tau_1 + g\chi(-A - A_1 + A^3 M_{22})\sin\tau_1 + \qquad (7.16)$$
$$+ A^3 M_{23}\cos 3\tau_1 + g\chi A^3 M_{24}\sin 3\tau_1.$$

Figures 5 and 6 give the results of several calculations for a square supported plate with an absolutely rigid rod assembly (Bibl.2). The divergence between the first and second approximations remains small so long as $u - u_*$ is

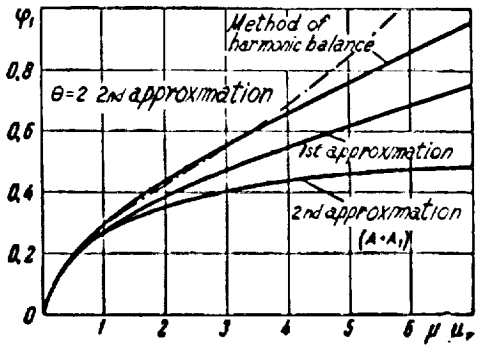

Fig.5

small and the dimensionless temperature $\vartheta < 3.5$.

8. The method of small parameters is applicable only in the region lying below the line BF (Fig.2), i.e., below the reduced temperature:

$$\theta = \frac{\omega_{11}^2 + \omega_{21}^2}{s_{11} + s_{21}} \quad . \tag{8.1}$$

At the point B, the generating frequency becomes zero, while above the value of ω_* it becomes complex. At the same value of θ, the correction for the amplitude of the generating solutions increases without limit. Further than that, application of the results obtained at small τ, i.e., near the line BE, /94 in a region remote from it, would be merely of formal character. To obtain results applicable over a wider region, let us apply the method of harmonic balance (Bibl.1).

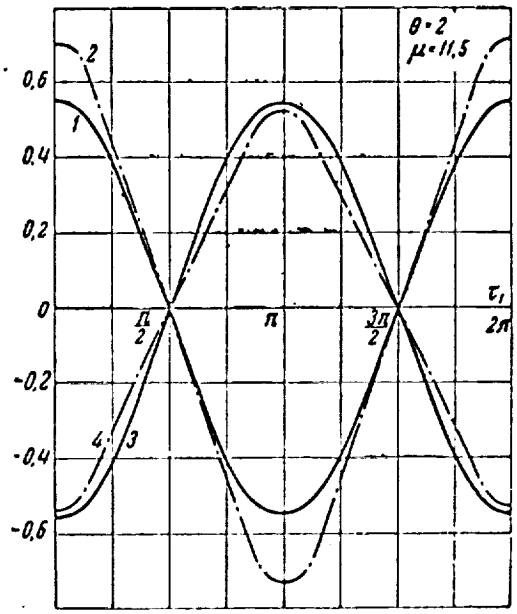

Fig.6

We will seek the solution of the system (2.5) in the form

$$\varphi_1 = A_1 \cos \omega\tau + B_1 \sin \omega\tau + \ldots, \quad \varphi_2 = A_2 \cos \omega\tau + \ldots, \tag{8.2}$$

where A_1, A_2, B_1, ω are unknowns to be determined. This choice of solutions is permissible, since we are considering steady flutter, i.e., we are seeking the limit cycle. Substituting eq.(8.2) into eqs.(2.5), we multiply each of the equations by cos $\omega\tau$ and by sin $\omega\tau$ and integrate from zero to 2π. This yields the system of four algebraic equations in four unknowns

$$\begin{aligned}
&-A_1\omega^2 + B_1 g\omega + \Omega_1^2 A_1 - \mu A_2 + c_{11}\left(\frac{3}{4}A_1^3 + \frac{3}{4}A_1 B_1^2\right) + \frac{3}{4}c_{12}A_1 A_2^2 = 0\\
&-B_1\omega^2 - A_1 g\omega + B_1\Omega_1^2 + c_{11}\left(\frac{3}{4}A_1^2 B_1 + \frac{3}{4}B_1^3\right) + c_{12}\frac{1}{4}B_1 A_2^2 = 0\\
&-A_2\omega^2 + \Omega_2^2 A_2 + \mu A_1 + c_{21}\left(\frac{3}{4}A_1^2 A_2 + \frac{1}{4}B_1^2 A_2\right) + \frac{3}{4}c_{22}A_2^3 = 0\\
&-A_2\omega g + \mu B_1 + \frac{1}{2}c_{21}A_1 B_1 A_2 = 0.
\end{aligned} \tag{8.3}$$

From the last equation, we have the relation

$$B_1 = \frac{g\omega A_2}{\mu + \frac{1}{2}c_{21}A_1 A_2}, \tag{8.4}$$

which we now substitute into the first three equations. Assuming the damping to be small, we neglect all terms containing g in the second and higher powers,

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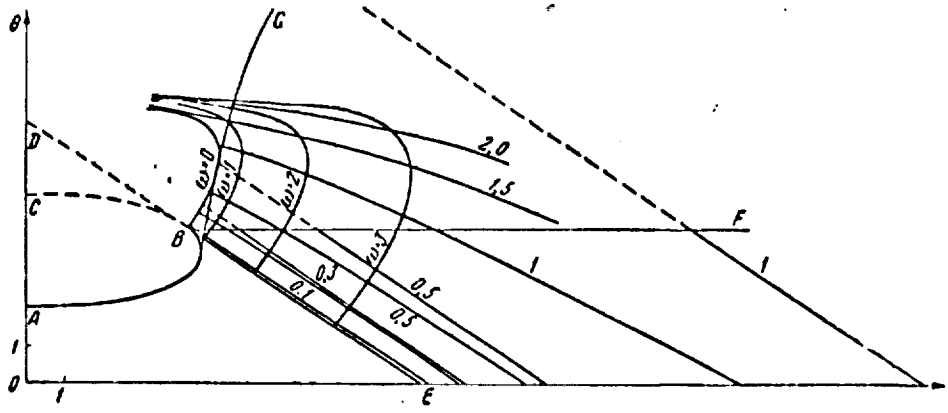

Fig.7

which substantially simplifies the system. Introducing, for convenience, the relation $\gamma = A_2/A_1$, we can obtain, from the simplified system,

$$A_1^2 = -\frac{2\mu(1-\gamma^2)}{\gamma(1+\gamma^2)c_{12}}, \tag{8.5}$$

$$A_1^2 = \frac{4}{3}\frac{\mu(1+\gamma^2)+\gamma(\Omega_2^2-\Omega_1^2)}{\gamma[c_{11}-c_{22}\gamma^2-c_{12}(1-\gamma^2)]}, \tag{8.6}$$

$$\omega^2 = \frac{\Omega_2^2+\Omega_1^2}{2} + \frac{A_1^2}{8}[3c_{11}+c_{12}+\gamma^2(3c_{22}+c_{12})]. \tag{8.7}$$

Eliminating A_1^2 from eqs.(8.5) and (8.6), we have

$$\frac{\Omega_2^2-\Omega_1^2}{\mu} = -\frac{\gamma^4(3c_{22}-c_{12})+\gamma^2(10c_{12}-3c_{11}-3c_{22})+3c_{11}-c_{12}}{2c_{12}\gamma(1+\gamma^2)}. \tag{8.8}$$

This equation, in the plane (μ, θ), defines a family of straight lines passing through a single point, each of which corresponds to some constant value of γ, i.e., the line of equal ratio of the amplitudes A_1 and A_2. For each specific value of γ and A_1 we can obtain μ and θ from eqs.(8.5) and (8.8). Then, by varying the value of γ and leaving A_1 constant, we obtain in the plane (μ, θ) the curve of A_1 = const. By giving A_1 other values we obtain a family of curves A_1 = const, after which the family A_2 = const is readily constructed. Considering further that γ and A_1 are known, we can find ω from eq.(8.7), and then B_1 from eq.(8.4).

The resultant solution, on the whole, is in satisfactory agreement with the solution of the method of small parameters, so long as the amplitude of the oscillations does not exceed 0.6 (Figs.5 and 7). This may be formally extended into the region above the line BF (Fig.7), but the question whether the solution in this region will be stable requires investigation. We note that Fyn (Bibl.5), applying an analogous method to the problem of the flutter of a rod (a plate under the conditions of cylindrical flexure), confines himself to the construction of solutions in the neighborhood of the point D (Fig.7). However, the mechanical sense of these solutions remains unknown, and their stability has not been proved. On the other hand, the periodic solutions found by us are obviously stable, at least in the region adjacent to the line BE. The question of the relation between the surface of steady flutter and the surface of static buck-

ling is of great interest. We note that if the line $\omega = 0$ is plotted on the surface of flutter constructed by the method of harmonic balance, then, provided that θ is not too great, this line will run very close to the boundary line /96 of the surface of buckling. Thus, it is not impossible that the exact surface of flutter osculates the surface of buckling along the line BG (Fig.8).

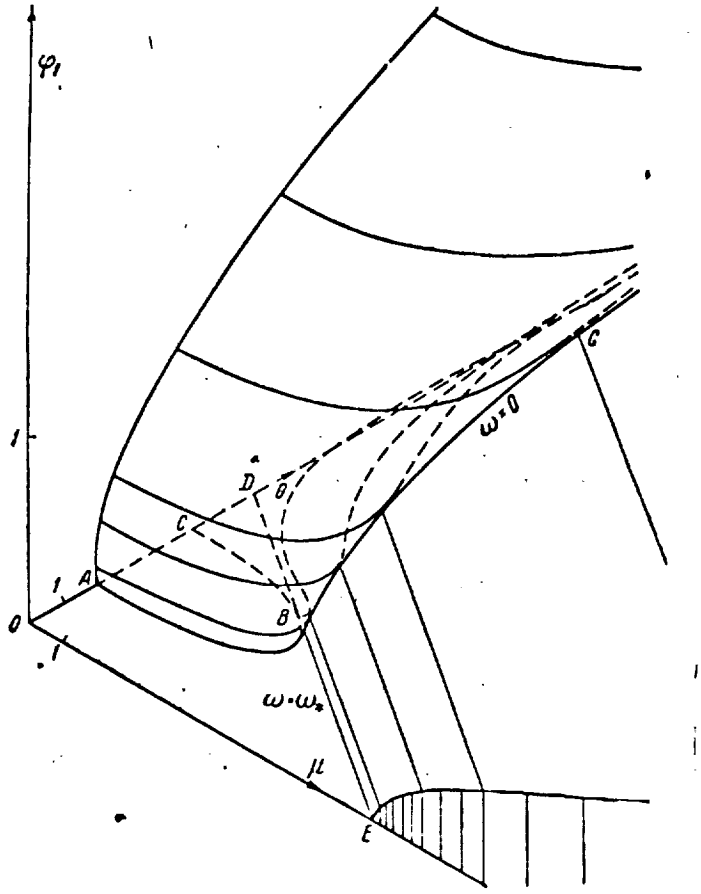

Fig.8

If this hypothesis holds, it will be relatively easy to systematize the various cases of the behavior of a panel under simultaneous variation of velocity and temperature. For example, if the circuit of the contour 1 - 12 - 1 is made in the plane (u, θ), as shown in Fig.2, we will obtain a transition from the stable plane form of the panel to the stable buckled form, followed by a

transition to steady flutter, and, finally, a return to the stable plane form.

BIBLIOGRAPHY

1. Bolotin, V.V., Gavrilov, Yu.V., Makarov, B.P., and Shveyko, Yu.Yu.: Nonlinear Problems of Stability of Plane Panels at High Supersonic Velocities (Nelineynyye zadachi ustoychivosti ploskikh paneley pri bol'shikh sverkhzvukovykh skorostyakh). Izv. Akad. Nauk SSSR, Otd. Tekhn. Nauk, Mekhan. i Mashinostr., No.3, 1959.

2. Bolotin, V.V.: Nonlinear Flutter of Plates and Shells (Nelineynyy flatter plastin i obolochek). Inzh. Sb., Vol.29, 1960.

3. Bolotin, V.V.: Thermal Buckling of Plates and Shells in a Supersonic Gas Flow (Temperaturnoye vypuchivaniye plastin i obolochek v sverkhzvukovom potoke gaza). Raschety na prochnost', No.6, Mashgiz, 1960.

4. Bolotin, V.V.: Nonconservative Problems of the Theory of Elastic Stability (Nekonservativnyye zadachi teorii uprugoy ustoychivosti). Fizmatgiz, 1961.

5. Fyn, Ya.Ts.: The Two-Dimensional Flutter of a Panel (O dvumernom flattere paneli). Sb. Mekhanika, No.1 (53), 1959.

Institute of Mechanics,
USSR Academy of Sciences.

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